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A_r-Weighted Poincaré-Type Inequalities for Differential Forms in Some Domains

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Abstract We prove local weighted integral inequalities for differential forms. Then by using the local results, we prove global weighted integral inequalities for differential forms in $L^s(\mu)$-averaging domains and in John domains, respectively, which can be considered as generalizations of the classical Poincaré-type inequality.

Keywords Differential forms, $L^s(\mu)$-averaging domains, Poincaré inequalities
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1 Introduction

Differential forms have wide applications in many fields, such as tensor analysis, potential theory, partial differential equations and quasiregular mappings, see [1–7]. Different versions of the classical Poincaré inequality have been established in the study of the Sobolev space and differential forms, see [2,6,8]. Susan G. Staples proves the Poincaré inequality in L^s-averaging domains in [8]. Tadeusz Iwaniec and Adam Lutoborski prove a local Poincaré-type inequality (see Lemma 2.2) in [6], which plays a crucial rule in generalizing the theory of Sobolev functions to differential forms. In this paper we prove local weighted Poincaré-type inequalities for differential forms in any kind of domains, and the global weighted Poincaré-type inequalities for differential forms in John domains and in $L^s(\mu)$-averaging domains, where μ is a measure defined by $d\mu = w(x)dx$ and $w \in A_r$. As we know, A-harmonic tensors are the special differential forms which are solutions to the A-harmonic equation for differential forms:

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$d^\star A(x,du)=0$, where $A:\Omega\times\wedge^l(\mathbf{R}^n)\to\wedge^l(\mathbf{R}^n)$ is an operator satisfying some conditions, see [5,6,9]. Thus, all of the results about differential forms in this paper remain true for A-harmonic tensors. Therefore, our new results concerning differential forms are of interest in some fields, such as those mentioned above.

Throughout this paper, we always assume Ω is a connected open subset of $\mathbf{R}^n$. Let $e_1,e_2,\ldots,e_n$ denote the standard unit basis of $\mathbf{R}^n$. For $l=0,1,\ldots,n$, the linear space of l-vectors, spanned by the exterior products $e_I=e_{i_1}\wedge e_{i_2}\wedge\cdots e_{i_l}$, corresponding to all ordered l-tuples $I=(i_1,i_2,\ldots,i_l)$, $1\le i_1<i_2<\cdots<i_l\le n$, is denoted by $\wedge^l=\wedge^l(\mathbf{R}^n)$. The Grassmann algebra $\wedge=\oplus\wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha=\sum\alpha^I e_I\in\wedge$ and $\beta=\sum\beta^I e_I\in\wedge$, the inner product in $\wedge$ is given by $\langle\alpha,\beta\rangle=\sum\alpha^I\beta^I$ with summation over all l-tuples $I=(i_1,i_2,\ldots,i_l)$ and all integers $l=0,1,\ldots,n$. We define the Hodge star operator $\star$: $\wedge\to\wedge$ by the rule $\star1=e_1\wedge e_2\wedge\cdots\wedge e_n$ and $\alpha\wedge\star\beta=\beta\wedge\star\alpha=\langle\alpha,\beta\rangle(\star1)$ for all α, $\beta\in\wedge$. Hence the norm of $\alpha\in\wedge$ is given by the formula $|\alpha|^2=\langle\alpha,\alpha\rangle=\star(\alpha\wedge\star\alpha)\in\wedge^0=\mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star:\ \wedge^l\to\wedge^{n-l}$ and $\star\star(-1)^{l(n-l)}:\ \wedge^l\to\wedge^l$. Letting $0<p<\infty$, we denote the weighted L^p-norm of a measurable function f over E by $||f||_{p,E,w}=\left(\int_E|f(x)|^p w(x)dx\right)^{1/p}$.

As we know, a differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbf{R}^n)$. In particular, for $l=0$, ω is a real function or a distribution. We denote the space of differential l-forms by $D'(\Omega,\wedge^l)$. We write $L^p(\Omega,\wedge^l)$ for the l-forms $\omega(x)=\sum_I\omega_I(x)dx_I=\sum\omega_{i_1i_2\cdots i_l}(x)dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_l}$ with $\omega_I\in L^p(\Omega,\mathbf{R})$ for all ordered l-tuples I. Thus $L^p(\Omega,\wedge^l)$ is a Banach space with norm $||\omega||_{p,\Omega}=\left(\int_\Omega|\omega(x)|^pdx\right)^{1/p}=\left(\int_\Omega(\sum_I|\omega_I(x)|^2)^{p/2}dx\right)^{1/p}$. Similarly, $W^1_p(\Omega,\wedge^l)$ are those differential l-forms on Ω whose coefficients are in $W^1_p(\Omega,\mathbf{R})$. The notations $W^1_{p,\rm loc}(\Omega,\mathbf{R})$ and $W^1_{p,\rm loc}(\Omega,\wedge^l)$ are self-explanatory. We denote the exterior derivative by $d:\ D'(\Omega,\wedge^l)\to D'(\Omega,\wedge^{l+1})$ for $l=0,1,\cdots,n$. Its formal adjoint operator $d^\star:\ D'(\Omega,\wedge^{l+1})\to D'(\Omega,\wedge^l)$ is given by $d^\star=(-1)^{nl+1}\star d\star$ on $D'(\Omega,\wedge^{l+1})$, $l=0,1,\cdots,n$.

We write $\mathbf{R}=\mathbf{R}^1$. Balls are denoted by B, and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B)=\sigma\operatorname{diam}(B)$. The n-dimensional Lebesgue measure of a set $E\subseteq\mathbf{R}^n$ is denoted by $|E|$. We call w a weight if $w\in L^1_{\rm loc}(\mathbf{R}^n)$ and $w>0$ a.e. Also in general $d\mu=wdx$ where w is a weight. The following result appears in [6]: Let $Q\subset\mathbf{R}^n$ be a cube or a ball. To each $y\in Q$ there corresponds a linear operator $K_y:C^\infty(Q,\wedge^l)\to C^\infty(Q,\wedge^{l-1})$ defined by $(K_y\omega)(x;\xi_1,\ldots,\xi_l)=\int_0^1t^{l-1}\omega(tx+y-ty;x-y,\xi_1,\cdots,\xi_{l-1})dt$ and the decomposition $\omega=d(K_y\omega)+K_y(d\omega)$.

We define another linear operator $T_Q:C^\infty(Q,\wedge^l)\to C^\infty(Q,\wedge^{l-1})$ by averaging K_y over all points y in Q: $T_Q\omega=\int_Q\varphi(y)K_y\omega dy$ where $\varphi\in C_0^\infty(Q)$ is normalized by $\int_Q\varphi(y)dy=1$. We define the l-form $\omega_Q\in D'(Q,\wedge^l)$ by $\omega_Q=|Q|^{-1}\int_Q\omega(y)dy,\ l=0,$ *and* $\omega_Q=d(T_Q\omega)$, $l=1,2,\ldots,n$, for all $\omega\in L^p(Q,\wedge^l)$, $1\le p<\infty$.

The following generalized Hölder's inequality will be used repeatedly.

Lemma 1.1 *Let $0<\alpha<\infty$, $0<\beta<\infty$ and $s^{-1}=\alpha^{-1}+\beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then $\| fg\|_{s,\Omega}\le\| f\|_{\alpha,\Omega}\cdot\| g\|_{\beta,\Omega}$ for any $\Omega\subset\mathbf{R}^n$.*

Definition 1.1 *We say the weight $w(x)>0$ satisfies the A_r-condition, where $r>1$, and* *write* $w \in A_r$ *if* $\sup_B(\frac{1}{|B|}\int_B w dx)(\frac{1}{|B|}\int_B w^{1/(1-r)}dx)^{r-1} < \infty$ *for any ball* $B \subset \mathbf{R}^n$.

We also need the following lemma [10].

Lemma 1.2 *If* $w \in A_r$, *then there exist constants* $\beta > 1$ *and* C, *independent of* w, *such that* $\| w \|_{\beta,Q} \leq C|Q|^{(1-\beta)/\beta} \| w \|_{1,Q}$ *for any cube or any ball* $Q \subset \mathbf{R}^n$.

2 Local Weighted Poincaré-Type Inequalities

The following version of the Poincaré inequality appears in [9].

Lemma 2.1 *Let* $u \in D'(Q, \wedge^l)$ *and* $du \in L^p(Q, \wedge^{l+1})$. *Then* $u - u_Q$ *is in* $W_p^1(Q, \wedge^l)$ *with* $1 < p < \infty$ *and* $\|u - u_Q\|_{p,Q} \leq C(n,p)|Q|^{1/n}\|du\|_{p,Q}$ *for* Q *a cube or a ball in* $\mathbf{R}^n$, $l = 0, 1, \ldots, n$.

T. Iwaniec and A. Lutoborski prove the following Poincaré-type inequality in [6].

Lemma 2.2 *Let* $u \in D'(Q, \wedge^l)$ *and* $du \in L^p(Q, \wedge^{l+1})$. *Then* $u - u_Q$ *is in* $L^{np/(n-p)}(Q, \wedge^l)$ *and* $(\int_Q |u - u_Q|^{np/(n-p)}dx)^{(n-p)/np} \leq C_p(n)(\int_Q |du|^p dx)^{1/p}$ *for* Q *a cube or a ball in* $\mathbf{R}^n$, $l = 0, 1, \ldots, n$ *and* $1 < p < n$.

We now prove the following version of the local weighted Poincaré-type inequality for differential forms.

Theorem 2.1 *Let* $u \in D'(B, \wedge^l)$ *and* $du \in L^p(B, \wedge^{l+1})$, *where* $1 < p < \infty$ *and* $l = 0, 1, \ldots, n$. *If* $w \in A_{1+\lambda}$ *for any* $\lambda > 0$, *then there exist constants* C, *independent of* u *and* du, *and* $\beta > 1$ *such that for any* α *with* $1 < \alpha < \beta$ *and* $(\alpha - 1)p > \beta$, *we have*

$$\left(\frac{1}{|B|}\int_B |u - u_B|^s w dx\right)^{1/s} \leq C|B|^{1/n}\left(\frac{1}{|B|}\int_B |du|^p w^{p/s} dx\right)^{1/p} \tag{2.1}$$

for all balls $B \subset \mathbf{R}^n$. *Here* $s = p(\alpha - 1)/\beta$.

Proof Since $w \in A_{1+\lambda}$, by Lemma 1.2, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\| w \|_{\beta,B} \leq C_1|B|^{(1-\beta)/\beta} \| w \|_{1,B} \tag{2.2}$$

for any cube or any ball $B \subset \mathbf{R}^n$. Choose $t = s\beta/(\beta - 1)$; then $1 < s < t$ and $\beta = t/(t - s)$. Since $1/s = 1/t + (t - s)/st$, by Hölder's inequality, Lemma 2.1 and (2.2), we have

$$\begin{aligned}\|u - u_B\|_{s,B,w} &= \left(\int_B \left(|u - u_B|w^{1/s}\right)^s dx\right)^{1/s} \\ &\leq \left(\int_B |u - u_B|^t dx\right)^{1/t} \left(\int_B \left(w^{1/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\ &= \|u - u_B\|_{t,B} \cdot \left(\int_B w^{t/(t-s)} dx\right)^{(t-s)/st}\end{aligned}$$

$$
\begin{aligned}
&\le C_2|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\cdot\|u-u_B\|_{t,B}\\
&\le C_2|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\cdot C_3|B|^{1/n}\|du\|_{t,B}\\
&= C_4|B|^{1/n}|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\cdot\|du\|_{t,B}.
\end{aligned}\tag{2.3}
$$

Now $t = s\beta/(\beta-1) < s\beta/(\alpha-1) = p$ and $1/t = 1/p + (p-t)/pt$, and by Hölder's inequality again we obtain

$$
\begin{aligned}
\|du\|_{t,B} &= \left(\int_B |du|^t dx\right)^{1/t} = \left(\int_B \left(|du|w^{1/s}w^{-1/s}\right)^t dx\right)^{1/t}\\
&\le \left(\int_B \left(|du|w^{1/s}\right)^p dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{pt/s(p-t)} dx\right)^{(p-t)/pt}\\
&= \left(\int_B |du|^p w^{p/s} dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{pt/s(p-t)} dx\right)^{(p-t)/pt}.
\end{aligned}\tag{2.4}
$$

Combining (2.3) and (2.4) yields

$$
\|u-u_B\|_{s,B,w} \le C_4|B|^{1/n+(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\cdot\left\|(1/w)^{1/s}\right\|_{pt/(p-t),B}\cdot\left(\int_B |du|^p w^{p/s}dx\right)^{1/p}. \tag{2.5}
$$

Choose $\lambda > 0$ such that $\lambda < 1 - \alpha/\beta$. Then $1+\lambda < 2-\alpha/\beta = r$. Hence $w \in A_{1+\lambda} \subset A_r$. By simple computation we know that $s(p-t)/pt = (2-\alpha/\beta) - 1 = r-1$. Thus, we have

$$
\begin{aligned}
&\|w\|_{1,B}^{1/s}\cdot\left\|(1/w)^{1/s}\right\|_{pt/(p-t),B}\\
&= \left(\int_B w dx\right)^{1/s}\left(\int_B \left(\frac{1}{w}\right)^{pt/s(p-t)} dx\right)^{(p-t)/pt}\\
&= \left(\left(\int_B w dx\right)\left(\int_B \left(\frac{1}{w}\right)^{pt/s(p-t)} dx\right)^{s(p-t)/pt}\right)^{1/s}\\
&= \left(|B|^{1+s(p-t)/pt}\right)^{1/s}\left(\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{pt/s(p-t)} dx\right)^{s(p-t)/pt}\right)^{1/s}\\
&= |B|^{1/s+1/t-1/p}\left(\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{1/s}\\
&\le C_5|B|^{1/s+1/t-1/p}.
\end{aligned}\tag{2.6}
$$

Substituting (2.6) in (2.5) implies

$$
\|u-u_B\|_{s,B,w} \le C_6|B|^{1/n+1/s-1/p}\left(\int_B |du|^p w^{p/s}dx\right)^{1/p}. \tag{2.7}
$$

We can write (2.7) as $(\frac{1}{|B|}\int_B |u-u_B|^s w(x)dx)^{1/s} \le C|B|^{1/n}(\frac{1}{|B|}\int_B |du|^p w^{p/s}dx)^{1/p}$. This completes the proof of Theorem 2.1.

We now prove another version of the local weighted Poincaré-type inequality for differential forms.

Theorem 2.2 *Let $u \in D'(B, \wedge^l)$ and $du \in L^n(B, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. If $1 < s < n$ and $w \in A_{n/s}$, then there exists a constant C, independent of u and du, such that*

$$\left(\frac{1}{|B|}\int_B |u-u_B|^s w^{s/n} dx\right)^{1/s} \le C\left(\int_B |du|^n w dx\right)^{1/n} \tag{2.8}$$

for any ball or any cube $B \subset \mathbf{R}^n$.

Proof Since $1/s = 1/n + (n-s)/ns$, by Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}\left(\int_B |u-u_B|^s w^{s/n} dx\right)^{1/s} &\le \left(\int_B \left(w^{1/n}\right)^n dx\right)^{1/n}\left(\int_B |u-u_B|^{ns/(n-s)} dx\right)^{(n-s)/ns} \\ &= \|w^{1/n}\|_{n,B} \cdot C_1 \|du\|_{s,B}.\end{aligned} \tag{2.9}$$

Using Hölder's inequality again, we have

$$\begin{aligned}\|du\|_{s,B} &= \left(\int_B \left(|du| w^{1/n} w^{-1/n}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B \left(|du| w^{1/n}\right)^n dx\right)^{1/n}\left(\int_B \left(\frac{1}{w}\right)^{s/(n-s)} dx\right)^{(n-s)/ns} \\ &= \left(\int_B |du|^n w dx\right)^{1/n} \cdot \left\|(1/w)^{1/n}\right\|_{ns/(n-s),B}.\end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\left(\int_B |u-u_B|^s w^{s/n} dx\right)^{1/s} \le C_1 \|w^{1/n}\|_{n,B} \cdot \left\|(1/w)^{1/n}\right\|_{ns/(n-s),B} \cdot \left(\int_B |du|^n w dx\right)^{1/n}. \tag{2.11}$$

Since $w \in A_{n/s}$, then

$$\begin{aligned}&\|w^{1/n}\|_{n,B} \cdot \left\|(1/w)^{1/n}\right\|_{ns/(n-s),B} \\ &= \left(\left(\int_B w dx\right)\left(\int_B \left(\frac{1}{w}\right)^{s/(n-s)} dx\right)^{(n-s)/s}\right)^{1/n} \\ &= \left(|B|^{n/s}\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(n/s-1)} dx\right)^{n/s-1}\right)^{1/n} \\ &\le C_2 |B|^{1/s}.\end{aligned} \tag{2.12}$$

Substituting (2.12) in (2.11), we obtain $\left(\int_B |u-u_B|^s w^{s/n} dx\right)^{1/s} \le C_3 |B|^{1/s}\left(\int_B |du|^n w dx\right)^{1/n}$, that is, $(\frac{1}{|B|}\int_B |u-u_B|^s w^{s/n} dx)^{1/s} \le C_3\left(\int_B |du|^n w dx\right)^{1/n}$. This completes the proof of Theorem 2.2.

3 Global Weighted Poincaré-Type Inequalities

S. G. Staples introduces the following L^s-averaging domains [8]: a proper subdomain $\Omega \subset \mathbf{R}^n$ is called an L^s-averaging domain, $s \geq 1$, if there exists a constant C such that $(\frac{1}{|\Omega|}\int_\Omega |u - u_\Omega|^s dm)^{1/s} \leq C \sup_{B\subset\Omega}(\frac{1}{|B|}\int_B |u - u_B|^s dm)^{1/s}$ for all $u \in L^s_{\text{loc}}(\Omega)$. Here $|\Omega|$ is the n-dimensional Lebesgue measure of Ω. Staples proves the Poincaré inequality in L^s-averaging domains in [8]. In [11], we introduce $L^s(\mu)$-averaging domains. We call a proper subdomain $\Omega \subset \mathbf{R}^n$ an $L^s(\mu)$-averaging domain, $s \geq 1$, if $\mu(\Omega) < \infty$ and there exists a constant C such that $(\frac{1}{\mu(B_0)}\int_\Omega |u - u_{B_0}|^s d\mu)^{1/s} \leq C \sup_{2B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u - u_B|^s d\mu\right)^{1/s}$ for some ball $B_0 \subset \Omega$ and all $u \in L^s_{\text{loc}}(\Omega; \wedge^l)$. Here the measure μ is defined by $d\mu = w(x)dx$, where $w(x)$ is a weight and $w(x) > 0$ a.e., and the supremum is over all balls $2B \subset \Omega$.

Now we prove the following global weighted Poincaré-type inequality in $L^s(\mu)$-averaging domains.

Theorem 3.1 *Let $u \in D'(\Omega, \wedge^l)$ and $du \in L^p(\Omega, \wedge^{l+1})$, where $n < p < \infty$ and $l = 0, 1, \ldots, n$. If $w \in A_{1+\lambda}$ for any $\lambda > 0$ and $w \geq \eta > 0$, then there exist constants C, independent of u and du, and $\beta > 1$ such that for any α with $1 < \alpha < \beta$ and $(\alpha - 1)p > \beta$, we have*

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega |u - u_{B_0}|^s w dx\right)^{1/s} \leq C\mu(\Omega)^{1/n}\left(\frac{1}{\mu(\Omega)}\int_\Omega |du|^p w^{p/s} dx\right)^{1/p} \tag{3.1}$$

for any $L^s(\mu)$-averaging domain Ω and some ball B_0 with $2B_0 \subset \Omega$. Here $s = (\alpha - 1)p/\beta$.

Proof By Theorem 2.1 and the definition of $L^s(\mu)$-averaging domains, we have

$$\begin{aligned}
\left(\frac{1}{\mu(\Omega)}\int_\Omega |u - u_{B_0}|^s d\mu\right)^{1/s} &\leq \left(\frac{1}{\mu(B_0)}\int_\Omega |u - u_{B_0}|^s d\mu\right)^{1/s} \\
&\leq C_1 \sup_{2B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u - u_B|^s d\mu\right)^{1/s} \\
&= C_1 \sup_{2B\subset\Omega}\left(\left(\frac{|B|}{\mu(B)}\right)^{1/s}\left(\frac{1}{|B|}\int_B |u - u_B|^s d\mu\right)^{1/s}\right) \\
&\leq C_1 \sup_{2B\subset\Omega}\left(\left(\frac{|B|}{\mu(B)}\right)^{1/s}\cdot C_2|B|^{1/n}\left(\frac{1}{|B|}\int_B |du|^p w^{p/s} dx\right)^{1/p}\right) \\
&\leq C_3 \sup_{2B\subset\Omega}\left(\mu(B)^{-1/s}|B|^{1/s+1/n-1/p}\left(\int_B |du|^p w^{p/s} dx\right)^{1/p}\right).
\end{aligned} \tag{3.2}$$

Noting that $\mu(B) = \int_B w dx \geq \int_B \eta dx = \eta|B|$, then

$$|B| \leq C_4\mu(B), \tag{3.3}$$

where $C_4 = 1/\eta$. Since $p > n$, then $1/n - 1/p > 0$, and from (3.3) we have

$$\begin{aligned}
\mu(B)^{-1/s}|B|^{1/s+1/n-1/p} &\leq \mu(B)^{-1/s}\cdot(C_4\mu(B))^{1/s+1/n-1/p} \\
&= C_5\mu(B)^{1/n-1/p} \leq C_5\mu(\Omega)^{1/n-1/p}.
\end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.2) yields

$$\begin{aligned}\left(\frac{1}{\mu(\Omega)}\int_\Omega |u-u_{B_0}|^s d\mu\right)^{1/s} &\le C_3 \sup_{2B\subset\Omega}\left(C_5\mu(\Omega)^{1/n-1/p}\left(\int_B |du|^p w^{p/s}dx\right)^{1/p}\right)\\ &\le C_6 \sup_{2B\subset\Omega}\left(\mu(\Omega)^{1/n-1/p}\left(\int_\Omega |du|^p w^{p/s}dx\right)^{1/p}\right)\\ &= C_6\mu(\Omega)^{1/n-1/p}\left(\int_\Omega |du|^p w^{p/s}dx\right)^{1/p}\\ &= C_6\mu(\Omega)^{1/n}\left(\frac{1}{\mu(\Omega)}\int_\Omega |du|^p w^{p/s}dx\right)^{1/p},\end{aligned}$$

that is, $(\frac{1}{\mu(\Omega)}\int_\Omega |u-u_{B_0}|^s w(x)dx)^{1/s} \le C\mu(\Omega)^{1/n}(\frac{1}{\mu(\Omega)}\int_\Omega |du|^p w^{p/s}dx)^{1/p}$. This completes the proof of Theorem 3.1.

Definition 3.1 *We call Ω, a proper subdomain of $\mathbf{R}^n$, a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined to any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that $d(\xi, \partial\Omega) \ge \delta|x-\xi|$ for each $\xi \in \gamma$. Here $d(\xi,\partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.*

As we know, John domains are bounded. Bounded quasiballs and bounded uniform domains are John domains. We also know that a δ-John domain has the following properties [9].

Lemma 3.1 *Let $\Omega \subset \mathbf{R}^n$ be a δ-John domain. Then there exists a covering $\mathcal{V}$ of Ω consisting of open cubes such that*

(i) $\sum_{Q\in\mathcal{V}} \chi_{\sigma Q}(x) \le N\chi_\Omega(x)$, $\sigma > 1$ and $x \in \mathbf{R}^n$,

(ii) *There is a distinguished cube $Q_0 \in \mathcal{V}$ (called the central cube) which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ such that for each $i = 0, 1, \ldots, k-1$, $Q \subset NQ_i$. There is a cube $R_i \subset \mathbf{R^n}$ (this cube does not need to be a member of $\mathcal{V}$) such that $R_i \subset Q_i \cap Q_{i+1}$, and $Q_i \cup Q_{i+1} \subset NR_i$.*

We also know that if $w \in A_r$, then the measure μ defined by $d\mu = w(x)dx$ is a doubling measure, that is, $\mu(2B) \le C\mu(B)$ for all balls B in $\mathbf{R}^n$, see [4, p. 299]. Since the doubling property implies $\mu(B) \approx \mu(Q)$ whenever Q is an open cube with $B \subset Q \subset \sqrt{n}B$, we may use cubes in place of balls whenever it is convenient to us.

Now we prove the following weighted global result in John domains.

Theorem 3.2 *Let $u \in D'(\Omega, \wedge^l)$ and $du \in L^n(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. If $1 < s < n$ and $w \in A_{n/s}$, then there exists a constant C, independent of u and du, such that*

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_Q|^s w^{s/n}dx\right)^{1/s} \le C\left(\int_\Omega |du|^n w dx\right)^{1/n}$$

for any δ-John domain $\Omega \subset \mathbf{R}^n$. Here Q is any cube in the covering $\mathcal{V}$ of Ω appearing in Lemma 3.1.

Proof We can write (2.8) as

$$\int_Q |u-u_Q|^s w^{s/n} dx \le C_1|Q| \left(\int_Q |du|^n w dx\right)^{s/n}, \tag{3.5}$$

where $Q \subset \mathbf{R}^n$ is any cube. Supposing $\sigma > 1$, by (3.5), and the condition (i) in Lemma 3.1, we have

$$\begin{aligned}\int_\Omega |u-u_Q|^s w^{s/n} dx &\le \sum_{Q\in\mathcal{V}} \int_Q |u-u_Q|^s w^{s/n} dx \le C_1 \sum_{Q\in\mathcal{V}} |Q| \left(\int_Q |du|^n w dx\right)^{s/n} \\ &\le C_1|\Omega| \sum_{Q\in\mathcal{V}} \left(\int_{\sigma Q} |du|^n w dx\right)^{s/n} \le C_1|\Omega| N \left(\int_\Omega |du|^n w dx\right)^{s/n} \\ &= C_2|\Omega| \left(\int_\Omega |du|^n w dx\right)^{s/n}.\end{aligned}$$

Thus, we have

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_Q|^s w^{s/n} dx\right)^{1/s} \le C_3 \left(\int_\Omega |du|^n w dx\right)^{1/n}.$$

This completes the proof of Theorem 3.2.

Remarks (1) Since $L^s(\mu)$-averaging domains reduce to L^s-averaging domains if $w = 1$ (so $d\mu = w(x)dx = dx$), then Theorem 3.1 also holds if $\Omega \subset \mathbf{R}^n$ is an L^s-averaging domain. (2) In [11] we proved that if Ω is a John domain and μ is a measure defined by $d\mu = w(x)dx$, where $w \in A_r$, then Ω is an $L^s(\mu)$-averaging domain. Therefore, Theorem 3.1 is also true if Ω is a John domain.

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